Reduced resistive MHD with general density I: Model and stability results

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Reduced resistive MHD with general density I: Model and stability results

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Abstract

The aim of this paper is to derive a general model for the reduced resistive MagnetoHydroDynamics and to study its mathematical structure. The model is established for arbitrary density profiles in the poloidal section of the toroidal geometry of Tokamaks. The stability of global weak solutions, on the one hand, and the stability of the fundamental mode around initial data, on the other hand, are investigated.

Keywords: Tokamaks, reduced MagnetoHydroDynamics, current hole.
AMS subject classification: 93A30, 35Q35, 76E25, 82D10

1 Introduction

Strong magnetic field are used to confine a plasma in Tokamaks, so that the conditions needed for thermonuclear fusion are reachable [9].

Reduced resistive magnetohydrodynamic (MHD) models have been proposed [5] to investigate a special type of instability appearing in Tokamaks, see Figure 1: this instability is called the "Current Hole". Essentially one observes that a stationary physical current profile becomes unstable and is replaced by a profile with almost zero amplitude, as described schematically in Figure 2. The "Current Hole" phenomena has been indirectly observed in JET [12] and JT-60 [10] and is a scenario for the ITER machine. It has therefore of major physical interest to better understand the Current Hole in view of the ITER project. In this work we focus on mathematical aspects of reduced resistive magnetohydrodynamic (MHD) models.

These models are obtained from a 2D simplification of full 3D MHD models with resistivity following the seminal work [13]: we refer to [18, 2] for cylindrical models and to [3, 4] for models in toroidal geometry. In all cases the unknowns are some scalar potentials which are defined in a cut

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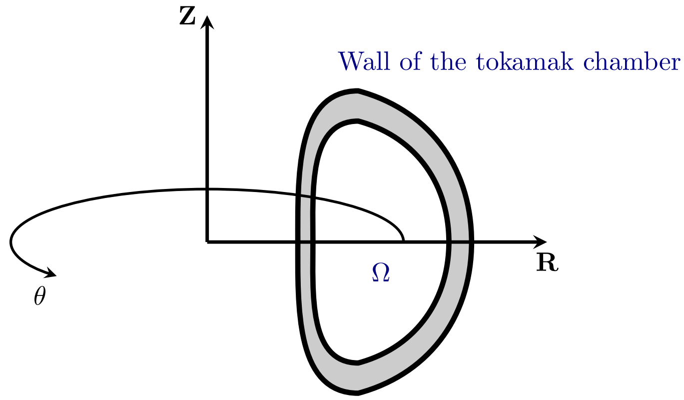

Figure 1: Schematic description of the poloidal section of a tokamak. The poloidal section is represented by the grey region plus the white region Ω. The main part of the plasma is assumed to be in the white region. The models developed in this work are defined in the 2D domain Ω.

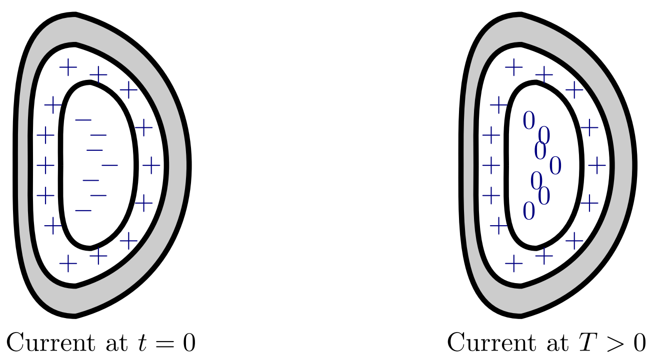

Figure 2: Principle of the Current Hole [5]. The exterior part of the grey region is still the wall of the tokamak chamber. At $t = 0$ the plasma current is negative inside the center region of the domain Ω and positive around. In some specific circumstances, it appears to be unstable: the negative internal current is replaced by an almost zero current.

of the initial 3D domain: the cut is planar in [18, 2] and it is a poloidal section of a torus in [3, 4]. Other generalized reduced MHD models may be found in [14].

The first aim of this work is to derive an original and more general reduced resistive MHD model. With respect to the usual reduced resistivity models [18, 2, 3, 4], we need less severe assumptions on the density profile, as it is explained in Figure 3. In our work the density profile is a general given function. To our knowledge all previous models are special cases of our model. In [18, 2] the density is constant and this means that the flow is assumed to be incompressible. In [3, 4] the density is scaled as R^{-2} and it corresponds to a flow in rotation and in inertial equilibrium (see Remark 4). Traditionally [3, 4], the family of 2D reduced MHD models is derived using an assumption of small curvature ($\varepsilon << 1$) and an assumption of small ratio of fluid pressure over magnetic pressure ($\beta << 1$). For the ITER project the curvature is moderate ($\varepsilon \approx 0.3$), so it is better to derive the model without using expansion with respect to ε: this is precisely what we do in Section 3, even if it is possible to recover the basic reduced MHD model as a limit of our model (see Remark 6). Since the fluid pressure does not show up in the final model, it means that we implicitly assume a small β regime.

The second aim is to study some mathematical properties of this general model: in this work we focus on the stability analysis of solutions because this issue seems to be important in order to establish a mathematical foundation for the simulations reported in [5, 7]. The model is endowed with an important energy identity, see (13) and further generalization. The stability results of this work are based on this energy estimate. Two types of stability are observed: stability of unsteady weak solutions and stability of particular stationary solutions which are constructed from the first eigenvector of the Grad-Shafranov operator.

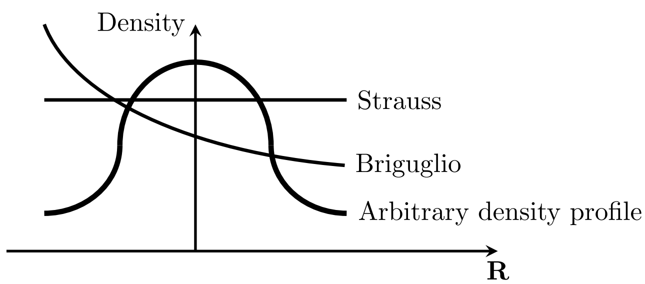

Figure 3: Cut of the density profile inside the Tokamaks chamber. The abscisse is $R \in [R^- R^+]$ with $0 < R^- < R^+$. The Strauss profile refers to the incompressible model [18, 2, 5]; the Briguglio profile refers to the ρR^2 constant profile used in [3, 4, 5]; the arbitrary profile corresponds to the new model proposed in this work. We believe that this profile is closer to the real situation in Tokamaks because the density of the plasma is higher in the core of the Tokamak.

2 Geometry and notations

The toroidal geometry of a generic tokamak is depicted in figure 1. The geometry is a torus obtained by the rotation of a poloidal section around the axis Z. We will use the cylindrical coordinates (R, θ, Z) which are related to the standard Cartesian coordinates (X, Y, Z) through

$$\begin{cases} X = R\cos\theta, \\ Y = R\sin\theta. \end{cases}$$

We assume that the interesting part of the plasma is confined in a two dimensional poloidal domain Ω (see Figure 1). It means that the plasma is confined in the three dimensional domain

$$(R, Z, \theta) \in \Omega \times [0, 2\pi[.$$

2.1 System of resistive MHD equations

The starting point of the modelling is the full system of resistive MHD equations

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \\ \partial_t \mathbf{B} = \nabla \wedge (\mathbf{u} \wedge \mathbf{B}) - \nabla \wedge (\eta \nabla \wedge \mathbf{B}), \\ \partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = \mathbf{J} \wedge \mathbf{B}, \quad \mathbf{J} = \nabla \wedge \mathbf{B}. \end{cases} \tag{1}$$

In the equation (1), ρ is the density, $\mathbf{u}$ is the velocity, $\mathbf{B}$ is the magnetic field and $\mathbf{J}$ is the current.

2.2 Reduced resistive model with general density

Some notations are introduced in this paragraph.

The **Poisson brackets** of two scalar functions is

$$[a, b] = \partial_R a \partial_Z b - \partial_Z a \partial_R b.$$

The **Grad-Shafranov operator** is defined by

$$\Delta^\star \psi = R\left(\partial_Z \left(\frac{\partial_Z \psi}{R}\right) + \partial_R \left(\frac{\partial_R \psi}{R}\right)\right) = \partial_{RR}\psi + \partial_{ZZ}\pi - \frac{1}{R}\partial_R \psi.$$

The **diffusion operator** Δ_ρ is defined by

$$\Delta_\rho \Phi = \rho R\left(\partial_Z \left(\frac{\partial_Z \psi}{\rho R}\right) + \partial_R \left(\frac{\partial_R \psi}{\rho R}\right)\right) = \Delta^\star \Phi - \frac{1}{\rho}(\partial_R \rho \partial_R \Phi + \partial_Z \rho \partial_Z \Phi).$$

Definition 1. Reduced resistivity MHD model with general density

The model writes

$$\begin{cases} \partial_t \psi = \frac{1}{\rho R}[\psi, \Phi] + \eta \Delta^\star \psi, \\ \partial_t \omega = \frac{1}{\rho R}[\omega, \Phi] - 2\frac{1}{(\rho R)^2}[\rho R, \Phi]\,\omega + \rho R\left[\psi, \frac{1}{\rho R^2}\Delta^\star \psi\right], \end{cases} \tag{2}$$

with $\Delta_\rho \Phi = \omega$. The domain is $(R, Z) \in \Omega$. The density profile is given. It is a time-independent and smooth function:

$$\rho \in W^{1,\infty}(\Omega), \quad 0 < \rho_- \le \rho \le \rho_+. \tag{3}$$

The resistivity is $\nu \ge 0$. The system is supplemented with natural Dirichlet boundary conditions

$$\psi = \Phi = 0 \text{ on } \partial\Omega.$$

The scalar potential ψ is the magnetic flux. The scalar potential Φ is the velocity potential and ρ is a given density profile. The magnetic field is derived from the magnetic flux

$$\mathbf{B} = \nabla\psi \wedge \nabla\theta. \tag{4}$$

Since $\nabla\theta = \frac{1}{R}\mathbf{e}_\theta$ then $\mathbf{B} = \nabla\psi \wedge \nabla\theta = \frac{1}{R}\nabla\psi \wedge \mathbf{e}_\theta$. Here the ∇ operator is defined in the X, Y, Z system of coordinates. By definition

$$\mathbf{B} = \nabla \wedge (\psi\nabla\theta) \quad \Longrightarrow \quad \nabla \cdot \mathbf{B} = 0.$$

And also

$$\mathbf{B} \cdot \mathbf{e}_\theta = 0$$

which means that the magnetic field is poloidal.

Remark 2. *In Section 5 we will consider another boundary condition, just for the simplicity of the mathematical analysis. In the numerical simulation of the Current Hole [5, 7] the numerical solution is negligible at the boundary so that the boundary conditions is of very little importance in this paper.*

Remark 3. *An important simplification has been made in (4). Indeed a more general representation formula [6] would be*

$$\mathbf{B} = F_0(t)\nabla\theta + \nabla\psi \wedge \nabla\theta$$

with a given $F_0(t)$. It means that the toroidal main magnetic field $F_0(t)\nabla\theta$ is neglected in (4). If one does not neglect this contribution, additional terms appear in the equations but we do not take it into account in this work. In some sense the source term J_c in (28) is a way to reintroduce this contribution.

Flows such that the density is unchanged correspond to $\nabla \cdot (\rho\mathbf{u}) = 0$. It is therefore convenient to assume that the velocity is represented in the form

$$\mathbf{u} = \frac{1}{\rho}\nabla\Phi \wedge \nabla\theta = \frac{1}{\rho R}\nabla\Phi \wedge \mathbf{e}_\theta$$

so that $\nabla \cdot (\rho\mathbf{u}) = 0$ holds true. Note that the velocity is also poloidal, that is $\mathbf{u} \cdot \mathbf{e}_\theta = 0$. One has by construction that

$$\partial_t\rho + \nabla \cdot (\rho\mathbf{u}) = 0.$$

That is why the density in (2) a given constant function $\rho(R, Z)$ and independent of the time variable.

3 Derivation of the model

This section is devoted to the derivation of (2) from (1) after convenient simplifications.

3.1 The magnetic equation

Consider the magnetic equation of (1)

$$\partial_t \mathbf{B} = \nabla \wedge (\mathbf{u} \wedge \mathbf{B}) - \nabla \wedge (\eta \nabla \wedge \mathbf{B}). \tag{5}$$

Since $\mathbf{B}$ is a rotational, we get

$$\partial_t \psi \nabla \theta = \mathbf{u} \wedge \mathbf{B} - \eta \nabla \wedge \mathbf{B} \ + \ \text{a gradient}$$

This gradient is set to zero and we will consider solutions of

$$\frac{1}{R} \partial_t \psi \mathbf{e}_\theta = \mathbf{u} \wedge \mathbf{B} - \eta \nabla \wedge \mathbf{B}.$$

Let us make two more hypotheses. Assume that the potentials are independent of the angle variables, that is

$$\psi = \psi(R, Z) \text{ and } \Phi = \Phi(R, Z).$$

Then

$$\mathbf{u} = \frac{1}{\rho R} \nabla \Phi \wedge \mathbf{e}_\theta = \frac{1}{\rho R} \left(-\partial_Z \Phi \mathbf{e}_R + \partial_R \Phi \mathbf{e}_Z \right)$$

and similarly

$$\mathbf{B} = \frac{1}{R} \left(-\partial_Z \psi \mathbf{e}_R + \partial_R \psi \mathbf{e}_Z \right),$$

so that

$$\mathbf{u} \wedge \mathbf{B} = \frac{1}{\rho R^2} \left(\partial_R \psi \partial_Z \Phi - \partial_Z \psi \partial_R \Phi \right) \mathbf{e}_\theta = \frac{1}{\rho R^2} [\psi, \Phi] \mathbf{e}_\theta.$$

To compute $\nabla \wedge \mathbf{B}$ we first notice that $\nabla \wedge \mathbf{e}_R = \nabla \wedge (\nabla R) = 0$ and that $\nabla \wedge \mathbf{e}_Z = 0$. Therefore

$$\nabla \wedge \mathbf{B} = \left(-\nabla \left(\frac{\partial_Z \psi}{R} \right) \wedge \mathbf{e}_R + \nabla \left(\frac{\partial_R \psi}{R} \right) \wedge \mathbf{e}_Z \right).$$

The first term is

$$\begin{aligned} \nabla \left(\frac{\partial_Z \psi}{R} \right) \wedge \mathbf{e}_R &= \left(\partial_R \left(\frac{\partial_Z \psi}{R} \right) \mathbf{e}_R + \partial_Z \left(\frac{\partial_Z \psi}{R} \right) \mathbf{e}_Z + \frac{1}{R} \partial_\theta \left(\frac{\partial_Z \psi}{R} \right) \mathbf{e}_\theta \right) \wedge \mathbf{e}_R \\ &= \partial_Z \left(\frac{\partial_Z \psi}{R} \right) \mathbf{e}_\theta. \end{aligned}$$

The second term is

$$\begin{aligned} \nabla \left(\frac{\partial_R \psi}{R} \right) \wedge \mathbf{e}_Z &= \left(\partial_R \left(\frac{\partial_R \psi}{R} \right) \mathbf{e}_R + \partial_Z \left(\frac{\partial_R \psi}{R} \right) \mathbf{e}_Z + \frac{1}{R} \partial_\theta \left(\frac{\partial_R \psi}{R} \right) \mathbf{e}_\theta \right) \wedge \mathbf{e}_Z \\ &= -\partial_R \left(\frac{\partial_R \psi}{R} \right) \mathbf{e}_\theta. \end{aligned}$$

So

$$\nabla \wedge \mathbf{B} = -\left(\partial_Z \left(\frac{\partial_Z \psi}{R} \right) + \partial_R \left(\frac{\partial_R \psi}{R} \right) \right) \mathbf{e}_\theta = -\frac{1}{R} \Delta^\star \psi \mathbf{e}_\theta.$$

The equation (5) writes

$$\frac{1}{R}\partial_t \psi \mathbf{e}_\theta = \frac{1}{\rho R^2}[\psi, \Phi]\mathbf{e}_\theta + \eta \frac{1}{R}\Delta^\star \psi \mathbf{e}_\theta.$$

After simplification we obtain the scalar equation

$$\partial_t \psi = \frac{1}{\rho R}[\psi, \Phi] + \eta \Delta^\star \psi. \tag{6}$$

3.2 The impulse equation

We start from the impulse equation

$$\partial_t \mathbf{u} + \nabla \mathbf{u}\ \mathbf{u} + \frac{1}{\rho}\nabla p = \frac{\mathbf{J} \wedge \mathbf{B}}{\rho}, \quad \mathbf{J} = \nabla \wedge \mathbf{B}.$$

Define the vectorial vorticity

$$\Omega = \nabla \wedge \mathbf{u}$$

with the equation

$$\partial_t \Omega + \nabla \wedge (\nabla \mathbf{u}\ \mathbf{u}) = \nabla \wedge \left(\frac{\mathbf{J} \wedge \mathbf{B}}{\rho} \right)$$

where we have assume either that ∇p is small with respect to all other terms (it is a low β assumption) or that the pressure is a function of the density so that $\frac{1}{\rho}\nabla p = \nabla q(\rho)$ has zero vorticity. One has

$$\begin{aligned}
\Omega &= \nabla \wedge \left(\frac{1}{\rho} \nabla \Phi \wedge \nabla \theta \right) \\
&= \frac{1}{\rho} \nabla \wedge (\nabla \Phi \wedge \nabla \theta) - \frac{1}{\rho^2} \nabla \rho \wedge (\nabla \Phi \wedge \nabla \theta) \\
&= \frac{1}{\rho} \left(-\frac{1}{R} \Delta^\star \Phi \mathbf{e}_\theta \right) - \frac{1}{\rho^2} \left((\nabla \rho \cdot \nabla \theta) \nabla \Phi - (\nabla \rho \cdot \nabla \Phi) \nabla \theta \right) \\
&= \left(-\frac{1}{\rho R} \Delta^\star \Phi + \frac{1}{\rho^2 R} (\nabla \rho \cdot \nabla \Phi) \right) \mathbf{e}_\theta.
\end{aligned}$$

The equation rewrites

$$\Omega = -\frac{1}{\rho R} \Delta_\rho \Phi \mathbf{e}_\theta.$$

Notice that by construction

$$\Delta_\rho \Phi = \rho R \left(\partial_R \left(\frac{1}{\rho R} \partial_R \Phi \right) + \partial_Z \left(\frac{1}{\rho R} \partial_Z \Phi \right) \right)$$

and

$$\Delta^\star = \Delta_{\rho \equiv 1}.$$

Next, since $(\nabla \psi \cdot \mathbf{e}_\theta) = 0$, the computation of the right hand side $\nabla \wedge (\frac{\mathbf{J} \wedge \mathbf{B}}{\rho})$ gives

$$\frac{\mathbf{J} \wedge \mathbf{B}}{\rho} = -\left(\frac{\Delta^\star \psi \mathbf{e}_\theta}{\rho R} \right) \wedge \left(\frac{\nabla \psi \wedge \mathbf{e}_\theta}{R} \right) = -\frac{1}{\rho R^2} \Delta^\star \psi \nabla \psi.$$

Therefore

$$\frac{\mathbf{J}\wedge\mathbf{B}}{\rho} = -\left(\frac{\Delta^\star\psi\partial_R\psi}{\rho R^2}\mathbf{e}_R + \frac{\Delta^\star\psi\partial_Z\psi}{\rho R^2}\mathbf{e}_Z\right),$$

and then

$$\begin{aligned}
\nabla\wedge\left(\frac{\mathbf{J}\wedge\mathbf{B}}{\rho}\right) &= -\left(\nabla\left(\frac{\Delta^\star\psi\partial_R\psi}{\rho R^2}\right)\wedge\mathbf{e}_R + \nabla\left(\frac{\Delta^\star\psi\partial_Z\psi}{\rho R^2}\right)\wedge\mathbf{e}_Z\right) \\
&= -\left(\partial_Z\left(\frac{\Delta^\star\psi\partial_R\psi}{\rho R^2}\right) - \partial_R\left(\frac{\Delta^\star\psi\partial_Z\psi}{\rho R^2}\right)\right)\mathbf{e}_\theta \\
&= -\left[\psi, \frac{1}{\rho R^2}\Delta^\star\psi\right]\mathbf{e}_\theta.
\end{aligned}$$

Finally we analyze $\nabla\wedge(\nabla\mathbf{u}\,\mathbf{u})$. One has

$$\mathbf{u} = -\frac{1}{\rho R}\partial_Z\Phi\mathbf{e}_R + \frac{1}{\rho R}\partial_R\Phi\mathbf{e}_Z = \alpha\mathbf{e}_R + \beta\mathbf{e}_Z$$

where we have set $\alpha = -\frac{1}{\rho R}\partial_Z\Phi$ and $\beta = \frac{1}{\rho R}\partial_R\Phi$. Therefore

$$\nabla\mathbf{u} = \alpha\nabla\mathbf{e}_R + \mathbf{e}_R\otimes\nabla\alpha + \mathbf{e}_Z\otimes\nabla\beta.$$

Noticing that $\nabla\mathbf{e}_R = \frac{1}{R}\mathbf{e}_\theta\otimes\mathbf{e}_\theta$, one obtains

$$\nabla\mathbf{u}\mathbf{u} = (\alpha\partial_R\beta + \beta\partial_Z\beta)\,\mathbf{e}_R + (\alpha\partial_R\alpha + \beta\partial_Z\alpha)\,\mathbf{e}_Z$$

and

$$\begin{aligned}
\nabla\wedge(\nabla\mathbf{u}\mathbf{u}) &= \nabla(\alpha\partial_R\beta + \beta\partial_Z\beta)\wedge\mathbf{e}_R + \nabla(\alpha\partial_R\alpha + \beta\partial_Z\alpha)\wedge\mathbf{e}_Z \\
&= (\partial_Z(\alpha\partial_R\beta + \beta\partial_Z\beta) - \partial_R(\alpha\partial_R\alpha + \beta\partial_Z\alpha))\,\mathbf{e}_\theta \\
&= A\mathbf{e}_\theta.
\end{aligned}$$

One has the identity

$$A = \alpha\partial_R(\partial_Z\alpha - \partial_R\beta) + \beta\partial_Z(\partial_Z\alpha - \partial_R\beta) + (\partial_R\alpha + \partial_Z\beta)(\partial_Z\alpha - \partial_R\beta).$$

By definition

$$\partial_Z\alpha - \partial_R\beta = -\partial_R\left(\frac{1}{\rho R}\partial_R\Phi\right) - \partial_Z\left(\frac{1}{\rho R}\partial_Z\Phi\right) = -\frac{1}{\rho R}\Delta_\rho\Phi.$$

One also has

$$\begin{aligned}
\partial_R\alpha + \partial_Z\beta &= -\partial_R\left(\frac{1}{\rho R}\partial_Z\Phi\mathbf{e}_R\right) + \partial_Z\left(\frac{1}{\rho R}\partial_R\Phi\mathbf{e}_R\right) \\
&= \frac{\partial_R(\rho R)}{(\rho R)^2}\partial_Z\Phi - \frac{\partial_Z(\rho R)}{(\rho R)^2}\partial_R\Phi \\
&= \frac{1}{(\rho R)^2}[\rho R, \Phi].
\end{aligned}$$

Therefore

$$\begin{aligned}
A &= \frac{\partial_Z\Phi}{\rho R}\partial_R\left(\frac{\Delta_\rho\Phi}{\rho R}\right) - \frac{\partial_R\Phi}{\rho R}\partial_Z\left(\frac{\Delta_\rho\Phi}{\rho R}\right) - \frac{1}{(\rho R)^3}[\rho R, \Phi] \\
&= \frac{1}{(\rho R)^2}[\Delta_\rho\Phi, \Phi] + 2\frac{1}{(\rho R)^3}[\rho R, \Phi]\,\Delta_\rho\Phi.
\end{aligned}$$

For convenience we define

$$\omega = \Delta_\rho \Phi.$$

The equation rewrites

$$-\frac{1}{\rho R}\partial_t \omega \mathbf{e}_\theta + \frac{1}{(\rho R)^2}\left[\omega, \Phi\right]\mathbf{e}_\theta + 2\frac{1}{(\rho R)^3}\left[\rho R, \Phi\right]\omega \mathbf{e}_\theta = -\left[\psi, \frac{1}{\rho R^2}\Delta^\star \psi\right]\mathbf{e}_\theta.$$

We finally deduce the scalar equation

$$\partial_t \omega = \frac{1}{\rho R}\left[\omega, \Phi\right] - 2\frac{1}{(\rho R)^2}\left[\rho R, \Phi\right]\omega + \rho R\left[\psi, \frac{1}{\rho R^2}\Delta^\star \psi\right]. \tag{7}$$

The reduced resistive MHD model (1) corresponds to (6) and (7).

Remark 4. Cylindrical geometry with a constant ρR^2.
It is described in [20, 5]. In our case it is sufficient to set

$$\rho = \frac{K}{R^2} \tag{8}$$

in the general model (6-7). It corresponds to a situation [16] where the plasma in the torus is in uniform rotation with a constant angular velocity ω_c. In this case the centrifugal acceleration is

$$\rho v^2 = \rho R^2 \omega_c^2.$$

If this centrifugal force is constant, then the plasma is in some kind of mechanical equilibrium. It corresponds precisely to (8).

3.3 Non dimensional system

It is convenient to write the equations (1) with non dimensional variables. One rescales the time

$$\tau = \frac{t}{T_0}.$$

The radius is $0 < R^- \leq R \leq R^+$. Define the small parameter

$$\varepsilon = \frac{R^+ - R^-}{R^+ + R^-}.$$

The parameter ε controls the curvature of the torus. If $\varepsilon = 0$ the torus degenerates to a infinite cylinder. Set

$$a = \frac{R^+ - R^-}{2} \text{ and } R_0 = \frac{R^+ + R^-}{2}.$$

This parameter a is the half small axis of the poloidal section of the chamber of the tokamak. Notice that

$$R_0 \varepsilon = a.$$

Since a is considered as a constant, it means that R_0 has a ε^{-1} dependency.

Proposition 5. *The non-dimensional version of (1) writes*

$$\left\{\begin{array}{l} \partial_\tau \psi = \frac{1}{\rho(1+\varepsilon x)}[\ \psi, \Phi] + \eta \Delta^\star \psi, \\ \partial_\tau \omega = \frac{1}{\rho(1+\varepsilon x)}[\omega, \Phi] - \frac{2}{\rho^2(1+\varepsilon x)^2}[\rho(1+\varepsilon x), \Phi]\omega + \rho(1+\varepsilon x)\left[\psi, \frac{1}{\rho(1+\varepsilon x)^2}\Delta^\star \psi\right]. \end{array}\right. \tag{9}$$

Proof. One has

$$\begin{cases} R = \frac{a}{\varepsilon}(1+\varepsilon x), & x \in [-1,1], \\ Z = ay, & y \in \left[-\frac{b}{a}, \frac{b}{a}\right]. \end{cases}$$

We use the variables (x, y) instead of (R, Z). The reference density is ρ_0 so that $\widehat{\rho}$ is the rescaled density

$$\rho = \rho_0 \widehat{\rho}.$$

We consider a reference velocity

$$u_0 = \frac{a}{T_0}$$

and a reference magnetic field

$$B_0^2 = \rho_0 u_0^2 = \frac{\rho_0 D^2}{T_0^2}.$$

We rescale the potentials

$$\begin{cases} \psi = \frac{B_0 a^2}{\varepsilon} \widehat{\psi}, \\ \Phi = \frac{u_0 \rho_0 a^2}{\varepsilon} \widehat{\Phi}. \end{cases}$$

The rescaled Poisson bracket corresponds to

$$\widehat{[u,v]} = \partial_x u \partial_y v - \partial_y u \partial_x v = a^2 [a, b].$$

The rescaled Grad-Shafranov operator is

$$\widehat{\Delta^\star} = \partial_{xx} + \partial_{yy} - \frac{\varepsilon}{1+\varepsilon x} \partial_x = a^2 \Delta^\star.$$

The rescaled density operator is

$$\widehat{\Delta_\rho} = a^2 \Delta_\rho.$$

We finally obtain, defining the rescaled resistivity $\eta = D u_0 \widehat{\eta}$, the model in non-dimensional variables

$$\begin{cases} \partial_\tau \widehat{\psi} = \frac{1}{\widehat{\rho}(1+\varepsilon x)} \widehat{[\widehat{\psi}, \widehat{\Phi}]} + \widehat{\eta} \widehat{\Delta^\star} \widehat{\psi}, \\ \partial_\tau \widehat{\omega} = \frac{1}{\widehat{\rho}(1+\varepsilon x)} \widehat{\left[\widehat{\omega}, \widehat{\Phi}\right]} - \frac{2}{\widehat{\rho}^2 (1+\varepsilon x)^2} \left[\widehat{\rho}\widehat{(1+\varepsilon x)}, \widehat{\Phi}\right] \widehat{\omega} + \widehat{\rho}(1+\varepsilon x) \left[\widehat{\psi}, \widehat{\frac{1}{\widehat{\rho}(1+\varepsilon x)^2}} \widehat{\Delta^\star} \widehat{\psi}\right]. \end{cases}$$

□

Remark 6. Planar geometry with incompressibility
We set $\rho = 1$ and $\varepsilon = 0$ in (9). The model corresponds to the seminal reduced resistive incompressible model [18] in a cylinder.

3.4 A general viscous model

From that point we are interested in proving some regularity results about the following system

$$\begin{cases} \partial_t \psi = \frac{1}{\rho R}[\psi, \Phi] + \eta \Delta^\star \psi, \\ \partial_t \omega = \rho R \left(\frac{1}{(\rho R)^2} [\omega, \Phi] + \left[\frac{1}{(\rho R)^2}, \Phi \right] \omega \right) + \rho R \left[\psi, \frac{1}{\rho R^2} \Delta^\star \psi \right] + \nu \Delta_\rho \omega, \end{cases} \tag{10}$$

with $\Delta_\rho \Phi = \omega$, with Dirichlet boundary conditions for all unknowns

$$\psi = \Phi = \frac{\partial \Phi}{\partial n} = 0 \text{ on } \partial\Omega, \tag{11}$$

and corresponding to the initial data

$$(\Phi, \psi)_{t=0} = (\Phi_0, \psi_0). \tag{12}$$

With respect to (2) we have added a viscous term $\nu \Delta_\rho \omega$ in the second equation. This term models the viscosity of the fluid. In the sequel we will study the mathematical consequences of this choice. In case of a constant ρR^2 (cf Remark 4), then $\Delta_\rho \omega$ is equal to the viscous operator considered in [5]. Further investigations are needed to establish the optimal expression of this viscous term. We have also made another modifications in the boundary condition which now contains an homogeneous Neumann condition for the variable Φ. This is for mathematical convenience because it slightly simplifies the analysis.

4 Identities

We quote several formal identities which are true for regular solutions of the preceding system (10-11).

4.1 Preservation of the total magnetic flux

Lemma 7. *Assume $\eta = \nu = 0$. Then regular solutions of (10-11) satisfy*

$$\frac{d}{dt} \int_\Omega \rho R \psi dR dZ = 0.$$

Proof. It comes from

$$\frac{d}{dt} \int_\Omega \rho R \psi = \int_\Omega [\psi, \Phi] = \int_\Omega (\partial_R(\psi \partial_Z \Phi) - \partial_Z(\psi \partial_R \Phi)) = \int_{\partial\Omega} \psi \partial_{tan} \Phi d\sigma = 0.$$

In this formula ∂_{tan} is the tangential derivative. The boundary integral vanishes thanks to the Dirichlet boundary condition. $\square$

4.2 Preservation of the cross-helicity

Lemma 8. *Assume $\eta = \nu = 0$. Then regular solutions of (10-11) satisfy*

$$\frac{d}{dt} \int_\Omega \frac{1}{\rho R} \psi \omega dR dZ = 0.$$

Proof. It comes from

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \frac{1}{\rho R}\psi\omega &= \int_\Omega \frac{1}{\rho R}(\omega\partial_t\psi + \psi\partial_t\omega) \\
&= \int_\Omega \frac{1}{(\rho R)^2}\omega[\psi,\Phi] + \int_\Omega \psi\left(\frac{1}{(\rho R)^2}[\omega,\Phi] + \left[\frac{1}{(\rho R)^2},\Phi\right]\omega\right) \\
&\quad + \int_\Omega \psi\left[\psi, \frac{1}{\rho R^2}\Delta^*\psi\right] \\
&= \int_\Omega \frac{1}{(\rho R)^2}[\omega\psi,\Phi] + \int_\Omega \left[\frac{1}{(\rho R)^2},\Phi\right]\omega\psi + \int_\Omega \left[\frac{\psi^2}{2}, \frac{1}{\rho R^2}\Delta^\star\psi\right] \\
&= 0
\end{aligned}$$

after integration and use of the Dirichlet boundary condition. □

4.3 The energy identity

This energy identity will have fundamental consequences in the sequel.

Proposition 9. *Assume $\eta \geq 0$ and $\nu \geq 0$. Then regular solutions of (10-11) satisfy*

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left(\frac{|\nabla\psi|^2}{R} + \frac{|\nabla\Phi|^2}{\rho R}\right) + \eta\int_\Omega \frac{|\Delta^*\psi|^2}{R} + \nu\int_\Omega \frac{|\Delta_\rho\Phi|^2}{\rho R} = 0 \qquad (13)$$

Proof.

- By multiplying the first equation of (10) by $\frac{\Delta^*\psi}{R}$, we get

$$\int_\Omega \partial_t\psi\frac{\Delta^*\psi}{R} = \int_\Omega \frac{1}{\rho R^2}[\psi,\Phi]\Delta^*\psi + \eta\int_\Omega \frac{|\Delta^*\psi|^2}{R}.$$

Integrating by parts and using properties of the Poisson brackets, we obtain

$$-\int_\Omega \partial_t\nabla\psi\cdot\left(\frac{\nabla\psi}{R}\right) = -\int_\Omega \left[\psi, \frac{1}{\rho R^2}\Delta^*\psi\right]\Phi + \eta\int_\Omega \frac{|\Delta^*\psi|^2}{R}$$

and then

$$-\frac{1}{2}\frac{d}{dt}\int_\Omega \frac{|\nabla\psi|^2}{R} = -\int_\Omega \left[\psi, \frac{1}{\rho R^2}\Delta^*\psi\right]\Phi + \eta\int_\Omega \frac{|\Delta^*\psi|^2}{R}. \qquad (14)$$

- By multiplying the second equation of (10) by $\frac{\Phi}{\rho R}$, we get

$$\begin{aligned}
\int_\Omega \partial_t\omega\frac{\Phi}{\rho R} &= \int_\Omega \frac{1}{(\rho R)^2}[\omega,\Phi]\Phi - \int_\Omega \frac{2}{(\rho R)^3}[\rho R,\Phi]\omega\Phi \\
&\quad + \int_\Omega \left[\psi, \frac{1}{\rho R^2}\Delta^*\psi\right]\Phi + \nu\int_\Omega \Delta_\rho\omega\frac{\Phi}{\rho R}.
\end{aligned}$$

By similar calculations, we successively get

$$\begin{aligned}
-\int_\Omega \partial_t\left(\frac{\nabla\Phi}{\rho R}\right)\nabla\Phi &= \int_\Omega \left[\Phi, \frac{1}{(\rho R)^2}\Phi\right]\omega - \int_\Omega \frac{2}{(\rho R)^3}[\rho R,\Phi]\omega\Phi \\
&\quad + \int_\Omega \left[\psi, \frac{1}{\rho R^2}\Delta^*\psi\right]\Phi - \nu\int_\Omega \nabla\omega\cdot\frac{\nabla\Phi}{\rho R}
\end{aligned}$$

and

$$
\begin{aligned}
-\frac{1}{2}\frac{d}{dt}\int_\Omega \frac{|\nabla\Phi|^2}{\rho R} &= \overbrace{\int_\Omega \left[\Phi, \frac{1}{(\rho R)^2}\right]\Phi\omega - \int_\Omega \frac{2}{(\rho R)^3}[\rho R, \Phi]\omega\Phi}^{=0} \\
&\quad + \int_\Omega \underbrace{[\Phi,\Phi]}_{=0}\frac{1}{(\rho R)^2}\omega + \int_\Omega \left[\psi, \frac{1}{\rho R^2}\Delta^*\psi\right]\Phi \\
&\quad +\nu\int_\Omega \omega\frac{\Delta_\rho\Phi}{\rho R} \\
&= \int_\Omega \left[\psi, \frac{1}{\rho R^2}\Delta^*\psi\right]\Phi + \nu\int_\Omega \frac{|\Delta_\rho\Phi|^2}{\rho R}. \qquad (15)
\end{aligned}
$$

- We conclude by summing (14) and (15).

□

5 Stability of weak solutions

In this section the stability of weak solutions is investigated. Our goal is to assess that variable density profiles are compatible with the standard theory of such systems for which we refer the reader to the seminal contributions [15, 21]. We refer to [11] for a modern presentation of the theory in the context of liquid metals. Essentially it amounts to showing that the *a priori* estimates (16) controls the continuity of the non linear terms of the general model. We will assume in this section that $\nu > 0$, $\eta > 0$, that (3) holds, and that $0 < R^- < R < R^+$.

5.1 Main result

Definition 10. *We define as a weak solution of the system (10-11), any couple of functions (ψ, Φ) such that*

- the following properties of regularity are satisfied:

$$\nabla\Phi, \nabla\psi \in L^\infty\left(0,T;L^2(\Omega)\right) \quad \textit{and} \quad \Delta\Phi, \Delta\psi \in L^2\left(0,T;L^2(\Omega)\right), \qquad (16)$$

- the system (10) holds in $\mathcal{D}'((0,T)\times\Omega)$, for any time $T>0$,

- the homogeneous Dirichlet boundary conditions (11) are satisfied in $\mathcal{D}'(\partial\Omega)$:

$$\Phi = \psi = \frac{\partial\Phi}{\partial n} = 0 \ \text{ on } \partial\Omega,$$

- the initial conditions (12) hold with the associated regularities

$$\psi_0, \Phi_0 \in H_0^1(\Omega).$$

Theorem 11. *Consider a suitable sequence of regular solutions $(\Phi^n, \psi^n)_{n\in\mathbb{N}}$ of equations (10-11) satisfying $\Phi^n_{|t=0} = \Phi_0 \in H_0^1(\Omega)$ and $\psi^n_{|t=0} = \psi_0 \in H_0^1(\Omega)$. Then this sequence converges to a weak solution (Φ,ψ) of (10-11) in the sense of Definition 10.*

5.2 Proof of theorem 11

We consider a sequence of regular solutions (ψ^n, Φ^n) of (10-11) associated to the initial data $(\psi_0^n, \Phi_0^n) \in H_0^1(\Omega) \times H_0^1(\Omega)$.

5.2.1 A priori estimates

Following the proof of Proposition 9, we notice that the boundary conditions (11) make all boundary terms, appearing in the integrations by parts, vanish. Then we can recall here the energy identity, integrated on $[0,t]$, $t > 0$, satisfied by (ψ^n, Φ^n):

$$\int_\Omega \left(\frac{|\nabla \psi^n|^2}{2R} + \frac{|\nabla \Phi^n|^2}{2\rho R} \right) + \eta \int_0^t \int_\Omega \frac{|\Delta^* \psi^n|^2}{R} + \nu \int_0^t \int_\Omega \frac{|\Delta_\rho \Phi^n|^2}{\rho R}$$
$$= \int_\Omega \left(\frac{|\nabla \psi_0^n|^2}{2R} + \frac{|\nabla \Phi_0^n|^2}{2\rho R} \right) < +\infty \tag{17}$$

Lemma 12. *Any sequence of solutions ψ^n and Φ^n of (10,11) satisfy*

$$\psi^n, \Phi^n \text{ bounded in } \mathcal{H} = L^\infty \left(0,T; H_0^1(\Omega)\right) \cap L^2 \left(0,T; H^2(\Omega)\right) \tag{18}$$

independently of n and for any time $T > 0$.

Moreover, for all $\xi \in]0,1[$, independently of n and for any time $T > 0$, we have

$$\nabla \psi^n, \nabla \Phi^n \text{ bounded in } \mathcal{K}_\xi = L^{\frac{2}{1-\xi}} \left(0,T; L^s(\Omega)\right), \text{ for all } s \in \left]2, \frac{2}{\xi}\right[, \tag{19}$$

Therefore, there exists functions (ψ, Φ) such that

$$\begin{aligned}
&\psi^n \rightharpoonup \psi \text{ and } \Phi^n \rightharpoonup \Phi \text{ weakly in } \mathcal{H}, \\
&\nabla \psi^n \rightharpoonup \nabla \psi \text{ and } \nabla \Phi^n \rightharpoonup \nabla \Phi \text{ weakly in } \mathcal{K}_\xi, \text{ for all } \xi \in]0,1[, \\
&\Delta^* \psi^n \rightharpoonup \Delta^* \psi \text{ and } \omega^n \rightharpoonup \omega \text{ weakly in } L^2 \left(0,T; L^2(\Omega)\right).
\end{aligned}$$

Proof. The bounds (18) are straightforward from (17), since ρ and R are bounded from above and from below and since the Poincaré inequality is guaranteed by the homogeneous Dirichlet boundary conditions.

The bounds (19) are obtained by interpolation. Indeed, the bounds (18), together with the Sobolev embedding in 2-dimension $H^1(\Omega) \subset L^p(\Omega)$ for all $1 < p < +\infty$, imply that $\nabla \psi_n$ and $\nabla \Phi_n$ are bounded in $L^\infty \left(0,T; L^2(\Omega)\right)$ and $L^2 \left(0,T; L^p(\Omega)\right)$ for all $1 < p < +\infty$. The conclusion comes from the embedding

$$L^\infty \left(0,T; L^2(\Omega)\right) \cap L^2 \left(0,T; L^p(\Omega)\right) \subset L^r \left(0,T; L^s(\Omega)\right)$$

for all r, s such that

$$\frac{1}{r} = \frac{1-\xi}{2}, \quad \frac{1}{s} = \frac{\xi}{2} + \frac{1-\xi}{p},$$

with any $\xi \in]0,1[$ and any $1 < p < +\infty$.

□

5.2.2 Compactness properties

In order to have some compactness properties, we want to use the following theorems.

Theorem 13. *([15], Theorem 5.1, p.58) Let B be a Banach space, and B_0 and B_1 be two reflexive Banach spaces. Assume $B_0 \subset B$ with compact injection, $B \subset B_1$ with continuous injection.*

Fix $T < +\infty$, $1 < p_0 < +\infty$, $1 < p_1 < +\infty$. Then the injection

$$\{v \in L^{p_0}(0,T;B_0);\ \partial_t v \in L^{p_1}(0,T;B_1)\} \subset L^{p_0}(0,T;B)$$

is compact.

Theorem 14. *([8], Corollary 2.1, p.29) Let $\overline{O} \subset R^M$ be compact and let X be a separable Banach space. Assume that $v_n : \overline{O} \to X^*$, $n = 1, 2, \ldots$ is a sequence of measurable functions such that ess $\sup_{y \in \overline{O}} \|v_n(y)\|_{X^*} \leq M$ uniformely in $n = 1, 2, \ldots$. Moreover let the family of (real) functions $\langle v_n, \Phi\rangle : y \mapsto \langle v_n(y), \Phi\rangle$, $y \in \overline{O}$, $n = 1, 2, \ldots$ be equi-continuous for any fixed Φ belonging to a dense subset in the space X.*

Then $v_n \in C\left(\overline{O}; X^*_{\text{weak}}\right)$ for any $n = 1, 2, \ldots$, and there exists $v \in C\left(\overline{O}; X^*_{\text{weak}}\right)$ such that

$$v_n \to v \text{ in } C\left(\overline{O}; X^*_{\text{weak}}\right) \text{ as } n \to \infty$$

passing to a subsequence as the case may be.

To do so, we have to find good bounds on the time derivative of the magnetic and velocity potentials.

For the magnetic potential:

$$\partial_t \psi^n = \underbrace{\frac{1}{\rho R}[\psi^n, \Phi^n]}_{S_1^n} + \underbrace{\eta \Delta^\star \psi^n}_{S_2^n}. \tag{20}$$

By (18), we know that S_2^n belongs to $L^2\left(0,T;L^2(\Omega)\right)$ and that $S_1^n = \frac{1}{\rho R}\nabla\psi^n \cdot \nabla^\perp \Phi^n$ belongs to $L^\infty\left(0,T;L^2(\Omega)\right) \times L^2\left(0,T;H_0^1(\Omega)\right)$, which is included in $L^2\left(0,T;L^s(\Omega)\right)$, for all $1 < s < 2$ (since $H_0^1(\Omega) \subset L^p(\Omega)$, for all $1 < p < +\infty$).

Through (20) and thinking that ρ and R are bounded from below, we can conclude that $\partial_t \psi^n$ is bounded in $L^2\left(0,T;L^s(\Omega)\right)$, for all $1 < s < 2$.

Then, using Theorem 13, we can insure that

$$\psi^n \to \psi \text{ in } L^2\left(0,T;W^{1,p}(\Omega)\right), \text{ for all } 1 < p < +\infty, \tag{21}$$

and, using Theorem 14, we also get

$$\psi^n \to \psi \text{ in } C\left([0,T];L^p(\Omega)\right), \text{ for all } 1 < p < +\infty, \tag{22}$$

thanks to the compact injection of $H_0^1(\Omega)$ in $L^p(\Omega)$, for any $1 < p < +\infty$.

For the velocity potential:

$$\partial_t \omega^n = \underbrace{\frac{1}{\rho R}\left[\omega^n, \Phi^n\right]}_{T_1^n} \underbrace{-2\frac{1}{(\rho R)^2}\left[\rho R, \Phi^n\right]\omega^n}_{T_2^n} + \underbrace{\rho R\left[\psi^n, \frac{1}{\rho R^2}\Delta^\star\psi^n\right]}_{T_3^n} + \underbrace{\nu \Delta_\rho \omega^n}_{T_4^n}. \tag{23}$$

Let's deal with the bounds on T_1^n, T_2^n, T_3^n and T_4^n. Notice that we will sometimes forget ρ and R coefficients which are bounded from above and from below.

We first can rewrite the Poisson brackets as

$$[a,b] = \operatorname{div}(a\nabla^\perp b), \tag{24}$$

where $\nabla^\perp = (-\partial_y, \partial_x)$.

On the one hand, we write

$$\begin{cases} T_1^n = \operatorname{div}\left(\dfrac{1}{\rho R}\Delta_\rho \Phi^n \nabla^\perp \Phi^n\right) + \Delta_\rho \Phi^n \nabla^\perp \Phi^n \cdot \nabla\left(\dfrac{1}{\rho R}\right), \\ T_2^n = \dfrac{1}{\rho R^2}\Delta_\rho \Phi^n \nabla^\perp \Phi^n \cdot \nabla(\rho R), \\ T_3^n = \operatorname{div}\left(\dfrac{1}{R}\Delta^\star \psi^n \nabla^\perp \psi^n\right) + \dfrac{1}{\rho R^2}\Delta^\star \psi^n \nabla^\perp \psi^n \cdot \nabla(\rho R). \end{cases}$$

We can say that $\Delta_\rho \Phi^n \nabla^\perp \Phi^n$ and $\Delta^\star \psi^n \nabla^\perp \psi^n$ belong to the product space $L^2\left(0,T;L^2(\Omega)\right) \times \mathcal{K}_\xi$, which is included in $L^{\frac{2}{2-\xi}}\left(0,T;L^s(\Omega)\right)$, for all $1 < s < \frac{2}{1+\xi}$, and for any $\xi \in]0,1[$.

Therefore, if there exists $q > 2$ such that $\nabla(\rho R)$ belongs to $L^q(\Omega)$, then there exists $r > 1$ such that T_2^n is bounded in $L^{\frac{2}{2-\xi}}\left(0,T;L^r(\Omega)\right)$, whereas T_1^n and T_3^n are bounded in $L^{\frac{2}{2-\xi}}\left(0,T;W^{-1,s}(\Omega)\right) + L^{\frac{2}{2-\xi}}\left(0,T;L^r(\Omega)\right)$, for all $1 < s < \frac{2}{1+\xi}$.

But, in 2-dimension, $L^r(\Omega) \subset H^{-1}(\Omega)$, for all $r > 1$, so we can summarize as follows: T_1^n, T_2^n and T_3^n are all bounded in $L^{\frac{2}{2-\xi}}\left(0,T;W^{-1,s}(\Omega)\right)$, for all $1 < s < \frac{2}{1+\xi}$.

On the other hand, since ω^n is bounded in $L^2\left(0,T;L^2(\Omega)\right)$, we conclude that T_4^n is bounded in $L^2\left(0,T;H^{-2}(\Omega)\right)$.

As a consequence, by the equation (23), the time derivative $\partial_t \omega^n$ is bounded in $L^2\left(0,T;H^{-2}(\Omega)\right)$, what allows us to use Theorem 13 to conclude

$$\omega^n \to \omega \text{ in } L^2\left(0,T;W^{-1,p}(\Omega)\right), \text{ for all } 1 < p < +\infty, \tag{25}$$

or, equivalently

$$\Phi^n \to \Phi \text{ in } L^2\left(0,T;W^{1,p}(\Omega)\right), \text{ for all } 1 < p < +\infty. \tag{26}$$

Moreover, using Theorem 14, we get

$$\Phi^n \to \Phi \text{ in } C\left([0,T];L^p(\Omega)\right), \text{ for all } 1 < p < +\infty, \tag{27}$$

thanks to the compact injection of $H_0^1(\Omega)$ in $L^p(\Omega)$, for any $1 < p < +\infty$.

5.2.3 Convergences

- *Equations* (10) -

The only difficulties concern the quadratic terms S_1^n, T_1^n, T_2^n and T_3^n. Since the Poisson brackets can be rewritten as in (24), we only need to get the convergence in the sense of distributions for the terms of "$a\nabla^\perp b$"-type.

• In order to pass to the limit in the first equation of (10), we just have to deal with the term $\psi^n \nabla^\perp \Phi^n$.

We know that ψ^n strongly converges to ψ in $C([0,T]; L^p(\Omega))$, for all $1 < p < +\infty$ and that $\nabla^\perp \Phi^n$ weakly converges to $\nabla^\perp \Phi$ in $\mathcal{K}_\xi$, for all $\xi \in]0,1[$, and this is enough to conclude the convergence for S_1^n:

$$\psi^n \nabla^\perp \Phi^n \to \psi \nabla^\perp \Phi \quad \text{in } \mathcal{D}'\left((0,T) \times \Omega\right).$$

• For the second equation of (10), three terms are concerned.

Since $\nabla^\perp \Phi^n$ strongly converges to $\nabla^\perp \Phi$ in $L^2(0,T; L^p(\Omega))$, for all $1 < p < +\infty$ and since ω^n weakly converges to ω in $L^2\left(0,T; L^2(\Omega)\right)$, we get the expected convergence for T_1^n:

$$\omega^n \nabla^\perp \Phi^n \to \omega \nabla^\perp \Phi \text{ in } \mathcal{D}'\left((0,T) \times \Omega\right).$$

Analogous arguments can be expressed for the convergence of T_3^n:

$$\Delta^* \psi^n \nabla^\perp \psi^n \to \Delta^* \psi \nabla^\perp \psi \text{ in } \mathcal{D}'\left((0,T) \times \Omega\right).$$

The term T_2^n is a little bit different, it can be written $T_2^n = \omega^n \nabla(\rho R) \cdot \nabla^\perp \Phi^n$. The weak convergence of ω^n to ω in $L^2\left(0,T; L^2(\Omega)\right)$ and the strong convergence of $\nabla^\perp \Phi^n$ to $\nabla^\perp \Phi$ in $\mathcal{K}_\xi$, for all $\xi \in]0,1[$, gives the convergence:

$$[\rho R, \Phi^n]\omega^n \to [\rho R, \Phi]\omega \text{ in } \mathcal{D}'\left((0,T) \times \Omega\right).$$

- *Boundary conditions* (11) -

By Lemma 12, we know that $\psi^n \rightharpoonup \psi$, $\Phi^n \rightharpoonup \Phi$ and $\nabla \Phi^n \rightharpoonup \nabla \Phi$ in $L^2\left(0,T; H^1(\Omega)\right)$. Since the trace operator $T : H^1(\Omega) \to L^2(\partial\Omega)$ is linear and continuous, it is also weakly continuous and then we can deduce that

$$\psi^n \rightharpoonup \psi, \quad \Phi^n \rightharpoonup \Phi, \quad \nabla \Phi^n \rightharpoonup \nabla \Phi \quad \text{in } L^2\left(0,T; L^2(\partial\Omega)\right),$$

what gives the convergence for the boundary conditions (11).

- *Initial conditions* (12) -

Through (22) and (27), we obtain the strong convergences $\psi^n(0) \to \psi(0)$ and $\psi^n(0) \to \psi(0)$ in $L^p(\Omega)$, for all $1 < p < +\infty$, so that the initial conditions (12) are satisfied.

6 Stability of stationary solutions

In this section we study the stability of stationary solutions. We believe that it is an appropriate mathematical formulation of the Current Hole. The ultimate goal is to determine what are the stationary solutions which are also stable solutions. It is possible to relax the assumption of stationarity, in this case the questions is to determine quasi-stationary and stable solutions. In what follows we focus on spectral stability of a special family of stationary solutions which correspond to eigenvectors of the Grad-Shafranov operator.

In order to fit with the model used in [5] and to obtain a more convenient mathematical formulation of the Current Hole, we consider the problem with a source term

$$\left\{\begin{array}{l}\partial_t\psi = \frac{1}{\rho R}[\psi,\Phi] + \eta\left(\Delta^\star\psi - J_c\right),\\ \partial_t\omega = \rho R\left(\frac{1}{(\rho R)^2}[\omega,\Phi] + \left[\frac{1}{(\rho R)^2},\Phi\right]\omega\right) + \rho R\left[\psi,\frac{1}{\rho R^2}\Delta^\star\psi\right],\end{array}\right. \tag{28}$$

plus Dirichlet boundary conditions. The source term J_c is a forcing non-ohmic current [5], see Remark 3. The initial conditions are denoted

$$\psi_0 = \psi(t=0), \quad \omega_0 = \omega(t=0).$$

We will use assume in this section that $\nu \geq 0$, $\eta \geq 0$, that (3) holds, and that $0 < R^- < R < R^+$.

6.1 Stationary solutions

In our context, a stationary solution is such that the velocity is zero, which turns into $\Phi_0 = \omega_0 = 0$. In this case

$$-\Delta^\star\psi_0 = J_c \text{ and } \omega_0 = 0.$$

Plugging in (28) one gets the relation

$$\left[\psi_0, \frac{1}{\rho R^2}\Delta^\star\psi_0\right] = 0.$$

Any ψ_0 such that $\frac{1}{\rho R^2}\Delta^\star\psi_0 = f(\psi_0)$ for a given function f satisfies this condition. In this work we assume a spectral dependence that is

$$-\frac{1}{\rho R^2}\Delta^\star\psi_0 = \lambda\psi_0.$$

Equivalently ψ_0 is solution to

$$\left\{\begin{array}{ll}-\partial_R\left(\frac{1}{R}\partial_R\psi_0\right) - \partial_Z\left(\frac{1}{R}\partial_Z\psi_0\right) = \lambda\rho R\psi_0, & x\in\Omega,\\ \psi_0 = 0, & x\in\partial\Omega.\end{array}\right.$$

This problem admits a symmetric and positive weak formulation in $H_0^1(\Omega)$, see [1]. We deduce that there exists a complete family $(u_i,\lambda_i)_{i\geq 0}$ of real eigenvectors $u_i \in H_0^1(\Omega)$ and real eigenvalues $\lambda_i \in \mathbb{R}$ such that

$$\left\{\begin{array}{ll}-\partial_R\left(\frac{1}{R}\partial_R u_i\right) - \partial_Z\left(\frac{1}{R}\partial_Z u_i\right) = \lambda_i\rho R u_i, & x\in\Omega,\\ u_i = 0, & x\in\partial\Omega,\end{array}\right.$$

with the ordering

$$0 < \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots.$$

The spectral gap is positive [1]

$$\mu = \lambda_1 - \lambda_0 > 0. \tag{29}$$

The eigenvectors are orthonormal for the weighted L^2 scalar product and for the weighted H_0^1 scalar product

$$\int_\Omega \rho R u_i u_j = \delta_{ij} \text{ and } \int_\Omega \frac{1}{R}\nabla u_i\cdot\nabla u_j = \lambda_i\delta_{ij}. \tag{30}$$

Definition 15. *All initial data*

$$(\psi_0, \omega_0) = \gamma(u_i, 0), \quad \forall i, \tag{31}$$

are stationary for the source term $J_c = \Delta^\star u_i$. We may call them spectral initial data.

Remark 16. *The eigenvalues and eigenvectors are continuous with respect to the coefficient of the problem which is ρ. Equivalently the eigenvalues and eigenvectors are continuous with respect to the function*

$$w = \rho R^2. \tag{32}$$

With (32), bounds of various quantities are obtained uniformly with respect to the density profile.

6.2 Stability in the case $i = 0$

As a preliminary remark we stress that the generalization of (13) writes

$$\frac{d}{dt}\int_\Omega \frac{|\nabla\psi|^2}{2R} + \frac{|\nabla\Phi|^2}{2\rho R} + \eta\int_\Omega \frac{(\Delta^\star\psi)^2}{R} + \nu\int_\Omega \frac{\omega^2}{\rho R} = \eta\int_\Omega \frac{1}{R} J_c \Delta^\star\psi \tag{33}$$

from which we can deduce from the Cauchy-Schwarz inequality that

$$\frac{d}{dt}\int_\Omega \frac{|\nabla\psi|^2}{2R} + \frac{|\nabla\Phi|^2}{2\rho R} \le \eta\int_\Omega \frac{1}{2R} J_c^2.$$

It implies that ψ and Φ remain bounded in the H_0^1 norm

$$\|\psi(t)\|^2_{H_0^1(\Omega)} + \|\Phi(t)\|^2_{H_0^1(\Omega)} \le C\left(\|\psi_0\|^2_{H_0^1(\Omega)} + t\|J_c\|^2_{L^2(\Omega)}\right). \tag{34}$$

This inequality does not imply nor that $\psi(t)$ remains close to its initial condition ψ_0, neither that $\Phi(t)$ remains close to zero. Our goal is here to obtain some inequalities that will explain precisely that $\psi(t)$ remains close to its initial condition ψ_0, and that $\Phi(t)$ remains close to zero. It will establish the stability of the corresponding initial data.

Let us define the coefficients α_n of expansion of ψ over the eigenvector basis

$$\psi(t) = \sum_{n\ge 0}\alpha_n(t)u_n, \quad \alpha_n(t) = \int_\Omega \rho R\psi(t)u_n.$$

We set $\gamma = \alpha_0(0)$. We study the differences

$$\overline{\psi} = \psi - \gamma u_0 \quad \text{and} \quad \overline{\omega} = \omega - 0 = \omega.$$

Similarly we define $\overline{\Phi} = \Phi - \Phi_0 = \Phi$. We also assume that $J_c = \gamma\Delta^\star u_0$.

Proposition 17. *One has the identity*

$$\frac{d}{dt}\int_\Omega \left(\frac{\left|\nabla\overline{\psi}\right|^2}{2R} - \lambda_0\rho R\overline{\psi}^2 + \frac{\left|\nabla\overline{\Phi}\right|^2}{2\rho R}\right) = -\eta\,\mathcal{I}(\overline{\psi}) - \nu\int_\Omega \frac{\overline{\omega}^2}{\rho R} \tag{35}$$

where we have defined

$$\mathcal{I}(\overline{\psi}) = \int_\Omega \left(\frac{\left(\Delta^\star\overline{\psi}\right)^2}{R} + \lambda_0\rho R\overline{\psi}\Delta^\star\overline{\psi}\right). \tag{36}$$

Proof. Simple algebra shows that

$$\begin{cases} \partial_t\overline{\psi} = & \frac{1}{\rho R}[\gamma u_0, \overline{\Phi}] \\ & +\frac{1}{\rho R}[\overline{\psi}, \overline{\Phi}] + \eta\Delta^\star\overline{\psi}, \\ \partial_t\overline{\omega} = & \rho R\left[\gamma u_0, \frac{1}{\rho R^2}\Delta^\star\overline{\psi}\right] + \rho R\left[\overline{\psi}, \frac{1}{\rho R^2}\Delta^\star\gamma u_0\right] \\ & +\rho R\left(\frac{1}{(\rho R)^2}\left[\overline{\omega}, \overline{\Phi}\right] + \left[\frac{1}{(\rho R)^2}, \overline{\Phi}\right]\overline{\omega}\right) + \rho R\left[\overline{\psi}, \frac{1}{\rho R^2}\Delta^\star\overline{\psi}\right]. \end{cases} \tag{37}$$

The source term J_c has been cancelled. The right hand sides are the sum of a linear term with respect to $\overline{\psi}$ and $\overline{\omega}$ (this term is written just after the sign =) and of a quadratic term (written on the next line). Concerning the quadratic terms, the structure is identical to the structure of the system (10). Multiplying $\partial_t\overline{\psi}$ by $-\frac{1}{R}\Delta^\star\overline{\psi}$ and $\partial_t\overline{\omega}$ by $-\frac{1}{\rho R}\overline{\Phi}$ and integrating by parts in the domain, the following energy relation can be deduced:

$$\frac{d}{dt}\int_\Omega \frac{\left|\nabla\overline{\psi}\right|^2}{2R} + \frac{\left|\nabla\overline{\Phi}\right|^2}{2\rho R}$$

$$\begin{aligned} = \quad & -\eta\int_\Omega \frac{\left(\Delta^\star\overline{\psi}\right)^2}{R} - \nu\int_\Omega \frac{\overline{\omega}^2}{\rho R} - \int_\Omega \left(\frac{1}{\rho R}[\gamma u_0, \overline{\Phi}]\right)\left(\frac{1}{R}\nabla^\star\overline{\psi}\right) \\ & - \int_\Omega \left(\rho R\left[\gamma u_0, \frac{1}{\rho R^2}\Delta^\star\overline{\psi}\right] + \rho R\left[\overline{\psi}, \frac{1}{\rho R^2}\Delta^\star\gamma u_0\right]\right)\left(\frac{1}{\rho R}\overline{\Phi}\right) \\ = \quad & -\eta\int_\Omega \frac{\left(\Delta^\star\overline{\psi}\right)^2}{R} - \nu\int_\Omega \frac{\overline{\omega}^2}{\rho R} - \int_\Omega [\gamma u_0, \overline{\Phi}]\frac{1}{\rho R^2}\nabla^\star\overline{\psi} \\ & + \int_\Omega \left[\gamma u_0, \frac{1}{\rho R^2}\Delta^\star\overline{\psi}\right]\overline{\Phi} - \int_\Omega \left[\overline{\psi}, \frac{1}{\rho R^2}\Delta^\star\gamma u_0\right]\overline{\Phi} \\ = \quad & -\eta\int_\Omega \frac{\left(\Delta^\star\overline{\psi}\right)^2}{R} + \lambda_0\int_\Omega \left[\overline{\psi}, \gamma u_0\right]\overline{\Phi} - \nu\int_\Omega \frac{\overline{\omega}^2}{\rho R} \\ = \quad & -\eta\int_\Omega \frac{\left(\Delta^\star\overline{\psi}\right)^2}{R} + \lambda_0\int_\Omega \left[\gamma u_0, \overline{\Phi}\right]\overline{\psi} - \nu\int_\Omega \frac{\overline{\omega}^2}{\rho R}. \end{aligned}$$

Next we eliminate $\left[\gamma u_0, \overline{\Phi}\right]$ with the first equation of the system rewritten as

$$[\gamma u_0, \overline{\Phi}] = \rho R\partial_t\overline{\psi} - [\overline{\psi}, \overline{\Phi}] - \eta\rho R\Delta^\star\overline{\psi},$$

so that

$$\begin{aligned} \int_\Omega \left[\gamma u_0, \overline{\Phi}\right]\overline{\psi} &= \int_\Omega \rho R\partial_t\frac{\overline{\psi}^2}{2} - \int_\Omega \left[\frac{\overline{\psi}^2}{2}, \overline{\Phi}\right] - \eta\int_\Omega \rho R(\Delta^\star\overline{\psi})\overline{\psi} \\ &= \int_\Omega \rho R\partial_t\frac{\overline{\psi}^2}{2} - \eta\int_\Omega \rho R(\Delta^\star\overline{\psi})\overline{\psi}. \end{aligned}$$

□

The remaining step consists in showing that the left hand side is a non negative quadratic form, and that the right hand side is controlled. To do so, let us define

$$\widehat{\psi} = \sum_{n\geq 1}\alpha_n u_n = \overline{\psi} - \alpha_0 u_0.$$

Proposition 18. *The left hand side of (35) controls the H_0^1 norm of $\widehat{\psi}$ and $\overline{\Phi}$. There exists a constant $C > 0$ such that*

$$\mu \left\| \widehat{\psi} \right\|^2_{H_0^1(\Omega)} + \left\| \overline{\Phi} \right\|^2_{H_0^1(\Omega)} \leq C \int_\Omega \left(\frac{\left| \nabla \overline{\psi} \right|^2}{R} - \lambda_0 \rho R \overline{\psi}^2 + \frac{\left| \nabla \overline{\Phi} \right|^2}{\rho R} \right), \quad (38)$$

where μ is the spectral gap (29).

Proof. It is sufficient to remark that

$$\int_\Omega \rho R \overline{\psi}^2 = \sum_{n \geq 0} \alpha_n^2 \quad \text{and} \quad \int_\Omega \frac{|\nabla \overline{\psi}|^2}{R} = \sum_{n \geq 0} \lambda_n \alpha_n^2.$$

Therefore

$$\int_\Omega \left(\frac{\left| \nabla \overline{\psi} \right|^2}{R} - \lambda_0 \rho R \overline{\psi}^2 \right) = \sum_{n \geq 1} \left(\lambda_n - \lambda_0 \right) \alpha_n^2 \geq \mu \sum_{n \geq 1} \alpha_n^2$$

controls the H_0^1 norm of $\widehat{\psi}$. □

Proposition 19. *Assume that the function w defined in (32) is constant. Then the right hand side of (35) is non positive.*

Proof. The hypothesis $\rho R = \frac{w}{R}$ has a major consequence. Indeed the contribution to be analyzed in the right hand side of (35) is $\mathcal{I}(\overline{\psi})$.

One has

$$\begin{aligned} \mathcal{I}(\overline{\psi}) &= \sum_{n,m \geq 0} \alpha_n \alpha_m \int_\Omega \lambda_n \lambda_m \frac{\rho^2 R^4}{R} u_n u_m - \lambda_0 \lambda_m \rho^2 R^3 u_n u_m \\ &= w \sum_{n,m \geq 0} \alpha_n \alpha_m \int_\Omega \left(\lambda_n \lambda_m - \lambda_0 \lambda_m \right) \rho^2 R^3 u_n u_m \end{aligned}$$

Because of the orthogonality relations (30), we obtain

$$\mathcal{I}(\overline{\psi}) = w \sum_n \alpha_n^2 \left(\lambda_n^2 - \lambda_0 \lambda_n \right). \quad (39)$$

Therefore $\mathcal{I}(\overline{\psi}) \geq 0$ unconditionally and the claim is proved. □

Next we do not consider anymore that w is a constant. The method of analysis consists nevertheless in comparing $\mathcal{I}(\overline{\psi})$ with a functional that can be decomposed as in (39). But we first establish technical results.

Proposition 20. *Let $T > 0$. The coefficient $\alpha_0(t)$ satisfies the estimate*

$$|\alpha_0(t)| \leq |\alpha_0(0)| + C \int_0^t \left\| \widehat{\psi}(s) \right\|_{H_0^1(\Omega)} ds \quad (40)$$

for some constant $C > 0$ and for all $t \leq T$.

Proof. Using the first equation of (37) one obtains

$$\alpha_0'(t) = \frac{d}{dt}\int_\Omega \rho R\overline{\psi}(t)u_0 = \int_\Omega [\gamma u_0, \overline{\Phi}]u_0 + \int_\Omega [\overline{\psi}, \overline{\Phi}]u_0 + \eta \int_\Omega \rho R \Delta^\star \overline{\psi} u_0.$$

The first integral vanishes: $\int_\Omega [\gamma u_0, \overline{\Phi}]u_0 = \gamma \int_\Omega [u_0, u_0]\overline{\Phi} = 0$. The second integral is

$$\int_\Omega [\overline{\psi}, \overline{\Phi}]u_0 = \int_\Omega [\overline{\psi} - \alpha_0(t)u_0, \overline{\Phi}]u_0 = \int_\Omega [\widehat{\psi}, \overline{\Phi}]u_0.$$

Using the energy identity (34) it is evident that $\overline{\Phi}$ is bounded in $H_0^1(\Omega)$. Therefore

$$\left| \int_\Omega [\overline{\psi}, \overline{\Phi}]u_0 \right| \le C \left\| \widehat{\psi} \right\|_{H_0^1(\Omega)}.$$

The third integral is

$$\int_\Omega \rho R \Delta^\star \overline{\psi} u_0 = \int_\Omega \rho R \Delta^\star \widehat{\psi} u_0 + \alpha_0(t) \int_\Omega \rho R (\Delta^\star u_0) u_0$$
$$= \int_\Omega \rho R \Delta^\star \widehat{\psi} u_0 - \alpha_0(t)\lambda_0 \int_\Omega \rho^2 R^3 u_0^2.$$

One also has after one integration by parts

$$\left| \int_\Omega \rho R \Delta^\star \widehat{\psi} u_0 \right| \le C \left\| \widehat{\psi} \right\|_{H_0^1(\Omega)}.$$

Therefore one has the Gronwall type relation $\alpha_0'(t) + \sigma\alpha_0(t) = q(t)$ with $\sigma = \lambda_0 \int_\Omega \rho^2 R^3 u_0^2 > 0$ and

$$|q(t)| \le C \left\| \widehat{\psi}(t) \right\|_{H_0^1(\Omega)}$$

for some universal constant $C > 0$. So

$$\alpha_0(t) = e^{-\sigma t}\alpha_0(0) + \int_0^t e^{-\sigma(t-s)} q(s) ds.$$

This proves the claim. $\square$

Lemma 21. (Gronwall lemma) *Let $t \mapsto f(t)$ be a smooth non negative function such that*

$$f(t) \le A + B\int_0^t f(s)ds + C\int_0^t \int_0^s f(r)dr, \quad A, B, C \ge 0.$$

Then there exists a constant $D > 0$ such that

$$f(t) \le \left(1 + \left(Bt + Ct^2\right) e^{Dt}\right) A. \tag{41}$$

Proof. Set $u(t) = \int_0^t \int_0^s f(r)dr$ and $v = u' + u$. Then $v' \le A + Dv$ with $D = \max(B+1, C)$. Therefore a classical Gronwall lemma shows that

$$v(t) \le e^{Dt}v(0) + A\int_0^t e^{D(t-s)}ds = \frac{e^{Dt} - 1}{D} A \le te^{Dt}A,$$

that is

$$u'(t) + u(t) \le te^{Dt}A \Longrightarrow u'(t) \le te^{Dt}A.$$

For $0 \le s \le t$ one has $u'(s) + u(s) \le \left(e^{Dt}A\right) s$. Once more, the Gronwall lemma between 0 and t shows that

$$u(t) \le e^{-t}u(0) + \left(e^{Dt}A\right) \int_0^t e^{s-t} s ds \le \left(e^{Dt}A\right) t^2.$$

Since $f(t) \le A + Bu'(t) + Cu(t)$, the result is proved. $\square$

Proposition 22. *There exists a second order polynomial*

$$x \mapsto p_t(x) = -C_1 \beta \left|\alpha_0(t)\right| x + C_2 \mu x^2 - C_3 \beta x^2 \tag{42}$$

where μ is the spectral gap, $\beta = \|w - w_-\|_{W^{1,\infty}(\Omega)}$, and the three constants C_1, C_2 and C_3 are positive, such that

$$\mathcal{I}(\overline{\psi}(t)) \ge p_t\left(\|\widehat{\psi}\|_{H^1(\Omega)}\right). \tag{43}$$

Proof. We first remark that $\overline{\psi}(t) = \widehat{\psi}(t) + \alpha_0(t)u_0$, so that

$$\begin{aligned}
\mathcal{I}(\overline{\psi}) &= \mathcal{I}(\widehat{\psi}) + \alpha_0(t)^2 \mathcal{I}(u_0) + \alpha_0(t) \int_\Omega \frac{2\Delta^\star u_0 \Delta^\star \widehat{\psi}}{R} + \lambda_0 \rho R (u_0 \Delta^\star \widehat{\psi} + \widehat{\psi} \Delta^\star u_0) \\
&= \mathcal{I}(\widehat{\psi}) + \alpha_0(t) \int_\Omega \lambda_0 \rho R(-u_0 \Delta^\star \widehat{\psi} + \widehat{\psi} \Delta^\star u_0)
\end{aligned}$$

after simplifications.
• The integral term is rewritten for convenience as

$$\begin{aligned}
&\int_\Omega \lambda_0 \frac{w}{R}(-u_0 \Delta^\star \widehat{\psi} + \widehat{\psi} \Delta^\star u_0) \\
= &\int_\Omega \lambda_0 \frac{w - w_-}{R}(-u_0 \Delta^\star \widehat{\psi} + \widehat{\psi} \Delta^\star u_0) + w_- \int_\Omega \frac{\lambda_0}{R}(-u_0 \Delta^\star \widehat{\psi} + \widehat{\psi} \Delta^\star u_0) \\
= &\int_\Omega \lambda_0 \frac{w - w_-}{R}(-u_0 \Delta^\star \widehat{\psi} + \widehat{\psi} \Delta^\star u_0).
\end{aligned}$$

Here w_- is any constant. Let us choose for convenience

$$w_- = \min_{(R,Z)\in\Omega} w(R, Z) > 0.$$

We also define

$$\beta = \|w - w_-\|_{W^{1,\infty}(\Omega)}.$$

Then

$$\left|\int_\Omega \lambda_0 \frac{w - w_-}{R}(-u_0 \Delta^\star \widehat{\psi} + \widehat{\psi} \Delta^\star u_0)\right| \le C_1 \beta \|\widehat{\psi}\|_{H^1(\Omega)}.$$

Thanks to the energy identity (34) one gets

$$\alpha_0(t) \int_\Omega \lambda_0 \rho R(-u_0 \Delta^\star \widehat{\psi} + \widehat{\psi} \Delta^\star u_0) \ge -C_1 \left|\alpha_0(t)\right| \beta \|\widehat{\psi}\|_{H^1(\Omega)}. \tag{44}$$

• Let us define the integral

$$\mathcal{J}(\widehat{\psi}) = \int_\Omega \frac{\left(\Delta^\star \widehat{\psi}\right)^2}{\rho R^3} + \frac{\lambda_0}{R}\widehat{\psi}\Delta^\star\widehat{\psi}.$$

Due to the orthogonality relations (30)

$$\mathcal{J}(\widehat{\psi}) = \sum_{n\geq 0} \alpha_n^2 \left(\lambda_n^2 - \lambda_0\lambda_n\right) \geq 0$$

holds without condition. Notice also that one has

$$\mathcal{J}(\widehat{\psi}) \geq (\lambda_1 - \lambda_0)\sum_{n\geq 0}\lambda_n\alpha_n^2 \geq C_2(\lambda_1-\lambda_0)\|\widehat{\psi}\|^2_{H^1(\Omega)}. \tag{45}$$

• Next,

$$\begin{aligned}
\mathcal{I}(\widehat{\psi}) &= \int_\Omega w\frac{\left(\Delta^\star \widehat{\psi}\right)^2}{\rho R^3} + w\frac{\lambda_0}{R}\widehat{\psi}\Delta^\star\widehat{\psi} \\
&= \int_\Omega \left((w-w_-)\frac{\left(\Delta^\star \widehat{\psi}\right)^2}{\rho R^3} + (w-w_-)\frac{\lambda_0}{R}\widehat{\psi}\Delta^\star\widehat{\psi}\right) + w_-\mathcal{J}(\widehat{\psi}).
\end{aligned}$$

Then

$$\mathcal{I}(\widehat{\psi}) \geq w_-\mathcal{J}(\widehat{\psi}) + \int_\Omega (w-w_-)\frac{\lambda_0}{R}\widehat{\psi}\Delta^\star\widehat{\psi}.$$

Thanks to (45) one has the lower bound

$$\mathcal{I}(\widehat{\psi}) \geq C_2(\lambda_1-\lambda_0)\|\widehat{\psi}\|^2_{H^1(\Omega)} + \int_\Omega (w-w_-)\frac{\lambda_0}{R}\widehat{\psi}\Delta^\star\widehat{\psi}. \tag{46}$$

After one integration by parts, it comes

$$\left|\int_\Omega (w-w_-)\frac{\lambda_0}{R}\widehat{\psi}\Delta^\star\widehat{\psi}\right| \leq C_3\|w-w_-\|_{W^{1,\infty}(\Omega)}\|\widehat{\psi}\|^2_{H^1(\Omega)} \quad (C_3>0).$$

Therefore one obtains

$$\begin{aligned}
\mathcal{I}(\overline{\psi}) \quad \geq \quad & -C_1\,|\alpha_0(t)|\,\|w-w_-\|_{W^{1,\infty}(\Omega)}\|\widehat{\psi}\|_{H^1(\Omega)} \\
& + C_2(\lambda_1-\lambda_0)\|\widehat{\psi}\|^2_{H^1(\Omega)} - C_3\|w-w_-\|_{W^{1,\infty}(\Omega)}\|\widehat{\psi}\|^2_{H^1(\Omega)}
\end{aligned}$$

or also $\mathcal{I}(\overline{\psi}) \geq p_t\left(\|\widehat{\psi}\|_{H^1_0(\Omega)}\right)$, and the proof is ended. □

Theorem 23. *Let $T>0$. There exists a constant $C>0$, depending on time, such that*

$$\left\|\overline{\psi}(t)\right\|_{H^1_0(\Omega)} + \left\|\overline{\Phi}(t)\right\|_{H^1_0(\Omega)} \leq C\left(\left\|\overline{\psi}(0)\right\|_{H^1_0(\Omega)} + \left\|\overline{\Phi}(0)\right\|_{H^1_0(\Omega)}\right). \tag{47}$$

More precisely,

(i) there exists constants $c_1, c_2 > 0$ such that (47) holds for all $t \leq T$ with

$$C = c_1(1 + te^{c_2 t}),$$

(ii) if the function w has small variation in the sense

$$\beta = \|w - w_-\|_{W^{1,\infty}(\Omega)} < c_0(\lambda_1 - \lambda_0) \quad (c_0 > 0), \tag{48}$$

then there exists constants $c_3, c_4 > 0$ such that (47) holds for all $t \leq T$ with

$$C = c_3(1 + t^3 e^{c_4 t}).$$

Proof.

(i) One can lower bound the negative part of p_t (42) as

$$p_t(x) \geq -M\left(\alpha_0(t)^2 + x^2\right), \quad M > 0.$$

Inserting in (35)-(36)-(43), one obtains

$$\frac{d}{dt}\left(\mu \left\|\widehat{\psi}(t)\right\|^2 + \|\overline{\Phi}(t)\|^2\right) \leq M\left(\left\|\widehat{\psi}(t)\right\|^2 + \alpha_0(t)^2\right). \tag{49}$$

Thanks to the basic inequality $(a+b)^2 \leq 2a^2 + 2b^2$ applied to (40) one obtains

$$\frac{d}{dt}\left(\mu \left\|\widehat{\psi}(t)\right\|^2 + \|\overline{\Phi}(t)\|^2\right) \leq M\left(\left\|\widehat{\psi}(t)\right\|^2 + 2\alpha_0(0)^2 + 2\left(\int_0^t \left\|\widehat{\psi}(s)\right\| ds\right)^2\right).$$

The Cauchy-Schwarz inequality implies that

$$\frac{d}{dt}\left(\mu \left\|\widehat{\psi}(t)\right\|^2 + \|\overline{\Phi}(t)\|^2\right) \leq M\left(\left\|\widehat{\psi}(t)\right\|^2 + 2\alpha_0(0)^2 + 2t\int_0^t \left\|\widehat{\psi}(s)\right\|^2 ds\right)$$

for $t < T$. After integration between 0 and t one gets

$$\begin{aligned} \mu \left\|\widehat{\psi}(t)\right\|^2 + \|\overline{\Phi}(t)\|^2 \quad &\leq \quad \left(\mu \left\|\widehat{\psi}(0)\right\|^2 + \|\overline{\Phi}(0)\|^2\right) + M\int_0^t \left\|\widehat{\psi}(s)\right\|^2 ds \\ &\quad + 2MT\alpha_0(0)^2 + 2Mt\int_0^t\int_0^s \left\|\widehat{\psi}(r)\right\|^2 drds \end{aligned} \tag{50}$$

We can simplify and get

$$\left\|\widehat{\psi}(t)\right\|^2 \leq a\left(\|\overline{\psi}(0)\|^2 + \|\overline{\Phi}(0)\|^2\right) + b\int_0^t \left\|\widehat{\psi}(s)\right\|^2 ds + ct\int_0^t\int_0^s \left\|\widehat{\psi}(r)\right\|^2 drds. \tag{51}$$

Lemma 21 applied to the inequality (51) with the particular choices $(A, B, C) = \left(a\left(\|\overline{\psi}(0)\|^2 + \|\overline{\Phi}(0)\|^2\right), b, ct\right)$ shows that

$$\left\|\widehat{\psi}(t)\right\|^2 \leq \left(1 + bte^{dt} + ct^3 e^{dt}\right) a\left(\|\overline{\psi}(0)\|^2 + \|\overline{\Phi}(0)\|^2\right), \quad d > 0, \tag{52}$$

and lead to (47).

Proposition (19) corresponds to $b = c = 0$. In this case the solution remains close to the initial data for all time $t > 0$. On the other hand the general result with $(b,c) \neq (0,0)$ shows the continuity in the sense of Hadamard in $H_0^1(\Omega)$, but with an exponential growth of the distance between the initial data and the solution. It is reasonable to infer that if the function w is non constant, but with small variations, then the solution will remain closer to the initial data, closer than what is predicted by the previous theorem. This is indeed the case.

(ii) For a small enough constant c_0, the dominant coefficient of $p_t(x)$ is positive: $C_2\mu - C_3\beta = C_4 > 0$. Therefore

$$\begin{aligned} p_t(x) &= C_4 x^2 - C_1\beta\,|\alpha_0(t)|\,x \\ &= C_4\left(x - \frac{C_1\beta\,|\alpha_0(t)|}{2C_4}\right)^2 - \frac{C_1^2\beta^2}{4C_4}\alpha_0(t)^2 \\ &\geq -M\alpha_0(t)^2. \end{aligned}$$

One obtains

$$\frac{d}{dt}\left(\mu\left\|\widehat{\psi}\right\|^2 + \left\|\overline{\Phi}\right\|^2\right) \leq M\alpha_0(t)^2 \tag{53}$$

which is a simplification of (49). So instead of (51) one gets

$$\left\|\widehat{\psi}(t)\right\|^2 \leq a\left(\left\|\overline{\psi}(0)\right\|^2 + \left\|\overline{\Phi}(0)\right\|^2\right) + ct\int_0^t\int_0^s \left\|\widehat{\psi}(r)\right\|^2 drds.$$

We now get (52) with $b = 0$. The rest of the proof is evident. $\square$

6.3 Discussion

The model that we have considered is quite general since the boundary and the density profile are arbitrary. It will be perhaps necessary to modify the model in the future to better fit with the physical setting of the problem, but the main ingredients are already there. The general model is equipped with an energy identity which shows that it fits into the standard formulation of such problems established after the seminal works of Lions [15] and Temam [21].

The results obtained in the previous section show that the fundamental mode is stable in the sense of Hadamard, and that we control the growth in time of the difference with the initial condition in the case the initial solution is close to the fundamental mode. If the function w is a constant, then $p_t(x) \geq 0$, and

$$\left\|\overline{\psi}(t)\right\|_{H_0^1(\Omega)} + \left\|\overline{\Phi}(t)\right\|_{H_0^1(\Omega)} \leq c_1\left(\left\|\overline{\psi}(0)\right\|_{H_0^1(\Omega)} + \left\|\overline{\Phi}(0)\right\|_{H_0^1(\Omega)}\right)$$

holds (this is also a consequence of Lemma 19). If compares with the results of the main theorem of this section, it indicates that the dependence in time of

$$\left\|\overline{\psi}(t)\right\|_{H_0^1(\Omega)} + \left\|\overline{\Phi}(t)\right\|_{H_0^1(\Omega)}$$

if more important if the function w is less flat. We do not know if these estimates are optimal. The basic inequality (38) plays a crucial role.

The fundamental mode u_0 is non negative (this is a basic result of the spectral theory of coercive operators like the Poisson operator, see

[1]). Considering the physical situation for the Current Hole which is described in Figure 2, one sees that the initial condition of the Current Hole has both signs, + and −, that is a mathematical formulation of the problem must consider at least the second mode u_1 as initial solution. It is an open problem nowadays.

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